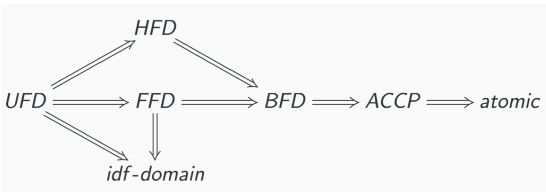
1 Overview

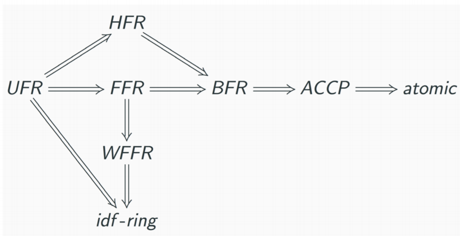
I am a commutative algebraist specializing in factorization theory, which is concerned with the decomposition of mathematical objects. In some sense, everyone is acquainted with factorization at an elementary level. Our introduction to the subject typically comes from learning how to decompose integers into the products of primes. Learning to decompose an integer into its smallest pieces, namely primes, gives us key insights into the integers as a whole. For instance, we can say that the integers have a property called unique factorization, which tells us that there is only one way to factor a given integer uniquely into the product of primes. If we formalize this notion, we can call prime numbers the atoms of the integers. In the same way that atoms form the building blocks of ordinary matter, mathematical atoms, or irreducible elements, form the building blocks of algebraic objects. We can glean insight into the characterization of algebraic properties of a mathematical object by understanding how that object breaks down into its smallest pieces, and by focusing on the interplay of its irreducible elements with one another. A consistent thread that permeates my research program is the investigation of the decomposition of mathematical objects in order to determine such an objects larger structural properties. The mathematical objects of my focus include commutative rings, monoids, and semigroups.

2 Commutative Algebra

Factorization with Zero Divisors My research program in commutative rings with zero divisors focuses on how factorization properties behave with respect to certain ring extensions. Of particular interest is the extension of a commutative ring with identity R to the polynomial ring $R[X]$. In this case we can ask, if one of these rings has a certain factorization property, does the other? The most ideal property is unique factorization, that is, a ring where every element factors uniquely into the product of atoms. Many familiar rings possess this property, such as $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}[x, y]$, and $\mathbb{Z}/4\mathbb{Z}$. The last example is a ring that contains *zero divisors*. A *zero divisor* is an element that can be multiplied by a nonzero element to give zero. A *domain* is a ring where the only zero divisor is 0. For instance in $\mathbb{Z}/4\mathbb{Z}$, $[2] \cdot [2] = [0]$. It is known that R is a unique factorization domain (UFD) if and only if $R[X]$ is a unique factorization domain, however if R is not a domain, that is if R is a ring that contains zero divisors, this is no longer true.



Property	UFD	HFD	FFD	idf	BFD	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes
R[X]	yes	no	yes	no	yes	no	no



Property	UFR	HFR	FFR	WFFR	idf	BFR	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes	yes
R[X]	no	no	no	no	no	no	no	no

Figure 1

There are many other properties weaker than unique factorization that can be studied. For instance, a half-factorial domain (HFD) is a domain in which every nonzero nonunit element has a factorization into atoms of the same length. A classic example is the ring $\mathbb{Z}[\sqrt{-5}]$, where we see that $6 = 3 \cdot 2 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ has two distinct factorizations into atoms but that each of these has the same number of factors, or length. Historically, the study of nonunique factorization was motivated by attempts to solve Fermat's last theorem, namely the incorrect assumption that if you adjoin a complex primitive root of unity to $\mathbb{Q}$, a UFD, the resulting ring is also a UFD. The class number of a ring gives a sort of measure of how far away a ring is from having unique factorization. A UFD has a class number of 1, whereas an HFD has a class number of 2. While number theorists were content to focus on these two factorization properties, [6] introduced a number of properties progressively weaker than unique factorization such as finite factorization (FF), bounded factorization (BF), ACCP, and atomic. Figure 1 highlights the relationship between certain factorization properties in the domain case and in the case where R is an arbitrary commutative ring. An arrow from one property to another means that the first property implies the second, for example if R has the unique factorization property it also has the half-factorial property. None of the arrows can be reversed. The tables in Figure 1 show the preservation of factorization properties under the extension to the polynomial ring. We see that if a domain R is a UFD, FFD, or BFD then $R[X]$ also inherits these respective properties. However in the case of an arbitrary ring R with zero divisors, the polynomial ring $R[X]$ does not inherit any of the factorization properties of the ring R.

In [3] we surveyed unique factorization in commutative rings with zero divisors, and gave a complete characterization of when a polynomial ring over an arbitrary commutative ring has some form unique factorization, and determined which forms of unique factorization of R extended to the polynomial ring $R[X]$. We also discussed irreducible elements of $R[X]$ and provided results on the factorization of powers of an indeterminate X over a commutative ring R. We were able to show that X is a product of irreducible elements (resp., principal primes) if and only if R is a finite direct product of indecomposable rings (resp., integral domains). In the case where X is a product of irreducibles, this factorization is unique up to order and associates while each power of X^n has unique factorization into irreducibles if and only if R is reduced and a finite direct product of indecomposable rings.

We can also consider factorization properties in more general settings. Monoid rings generalize both monoid domains and polynomial rings with zero divisors. Let R be a ring and S a monoid. We write the elements of the *monoid ring* $R[S]$ as "polynomials" with coefficients in R and exponents in S. It is possible for $R[S]$ to have zero divisors, even if R is a domain. For example if we take $R = \mathbb{Z}$ and $S = \mathbb{Z}/2\mathbb{Z}$, then in $\mathbb{Z}[S]$ we have that $(x+1)(x-1) = x^2 - 1 = 1 - 1 = 0$. Factorization in $R[S]$ is somewhat well known, but is relatively unexplored in the case where R can possibly contain zero divisors. In [14] we studied associate relations and the resulting notions of irreducibility and factorization length in monoid rings with zero divisors. We characterized monoid rings that are bounded factorization rings (BFRs) or satisfy the ascending chain condition on principal ideals (ACCP). We also characterized the monoid rings that are présimplifiable or domainlike. Many of these results concerned monoid rings $R[S]$ where S was torsion-free cancellative.

Numerical Semigroup Algebras A numerical semigroup is a subset $S \subseteq \mathbb{N}$ that is closed under addition, has finite complement in $\mathbb{N}$, and contains zero. Every numerical semigroup admits a unique generating set, and when we write $S = \langle n_1, \ldots, n_k \rangle = \{a_1 n_1 + \cdots + a_k n_k : a_i \in \mathbb{N}\}$ we assume $n_1, \ldots, n_k$ are minimal generators of S. The smallest nontrivial numerical semigroup is $S = \langle 2, 3 \rangle$, which includes all linear combinations of the generators 2 and 3, for example, $23 \in S$ since $23 = 2 \cdot 4 + 3 \cdot 5$. In this case S includes all natural numbers except 1. We call a a *gap* of a numerical semigroup if $a \notin S$, and the largest gap is called the *Frobenius number* of S. In the case $S = \langle 2, 3 \rangle$, the Frobenius number $F(S) = 1$. Given a numerical semigroup S and a field $\mathbb{F}$, we can form the *numerical semigroup algebra* $\mathbb{F}[S] = \{a_0 + a_1 x + \cdots + a_d x^d \in \mathbb{F}[x] : a_i = 0 \text{ whenever } i \notin S\} \subseteq \mathbb{F}[x]$ which consists only of terms x^i with $i \in S$. For example, the numerical semigroup algebra $\mathbb{F}_2[S]$ where $S = \langle 2, 3 \rangle$ does not contain linear polynomials since $1 \notin S$. Semigroup algebras, when viewed as quotients of toric ideals, are central to combinatorial commutative algebra, and arise in a host of statistical and computational applications.

In [9] we initiated the study of atomic density in numerical semigroup algebras, which is the limiting value of the ratio of irreducibles of degree n to total polynomials of degree n for some numerical semigroup algebra over a finite field $\mathbb{F}_q[S]$. Numerical semigroup algebras exhibit non unique factorization, and in $\mathbb{F}_q[S]$ you obtain many elements that would not be irreducible in $\mathbb{F}_q[x]$. For example, $x^2 + x^3$ is irreducible in $\mathbb{F}_2[S]$ with $S = \langle 2, 3 \rangle$ but not in $\mathbb{F}_2[x]$. Thus it is of interest to see what is happening to the number of irreducibles of a given degree n as n grows larger. The main result of [9] is that any numerical semigroup algebra $\mathbb{F}_q[S]$ has atomic density 0, that is

$$\lim_{n \to \infty} \rho_q^S(n) = 0 \qquad \text{with} \qquad \rho_q^S(n) = \frac{a_q^S(n)}{|\mathbb{F}_q[S]^{(n)}|} \qquad (*),$$

where $a_q^S(n)$ is the number of degree n irreducibles in $\mathbb{F}_q[S]$ and $|\mathbb{F}_q[S]|$ is the total number of degree n polynomials in $\mathbb{F}_q[S]$. This means that for a given numerical semigroup algebra S over a finite field, in some sense, 'most' polynomials in $\mathbb{F}_q[S]$ are reducible.

We were able to prove this main result by splitting the irreducible elements of $\mathbb{F}_q[S]$ into finitely many classes based on their factorizations in $\mathbb{F}_q[x]$, providing a bound on their factorization length that only depended on q and S, and then showing that the proportion of irreducibles in $\mathbb{F}_q[S]$ of degree n shrinks with n. We obtained more refined results for the semigroup algebra $\mathbb{F}_2[x^2, x^3]$ by proving that the irreducibles of $\mathbb{F}_2[x^2, x^3]$ can be partitioned into three distinct classes, and used these classes to provide a convergence rate on the limit $(*)$ above. We also used the classification of irreducibles to provide a formula for the number of irreducibles of each degree in $\mathbb{F}_2[x^2, x^3]$ in terms of the Möbius function, analagous to a well-known formula for the number of irreducible polynomials of each degree in $\mathbb{F}_q[x]$, that follows from the Möbius inversion formula. We continue this work by considering the following: **(1)** Can we determine the atomic densities of power series rings over numerical semigroup algebras? **(2)** Can we classify the type of irreducible polynomials in general numerical semigroup algebras, and provide a tighter bound on the number of irreducible polynomials of each degree therein?

Amicable Pairs *Amicable pairs*, or *friendly numbers*, in the natural numbers are a pair of distinct positive integers (m, n) where each integer is the sum of the proper divisors of the other integer. Thus, (m, n) is an amicable pair when $\sigma(m) = m + n = \sigma(n)$. The smallest amicable pair in $\mathbb{N}$ is the pair $(220, 284)$. In this case we see that the proper divisors of 220–1,2,4,5,10,11,20,22,44,55,110, sum to 284, and the proper divisors of 284–1,2,4,71,142, sum to 220. This pair was known by the Greek as early as 300 BCE, and until 1750 there were only three known amicable pairs. In 1750, Euler discovered 58 more amicable pairs by considering pairs of a certain form. This set the stage for classifying amicable pairs according to type based on the factorization of each number in the pair. For example, since $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$, the pair (220,284) is called a (2,1) pair, as 220 has two factors, and 284 has one factor outside of their common factor of 2^2.

There are now more than 1.2 billion known amicable pairs. The characterization of amicable pairs by type allows for searches for pairs using computer algorithms. It is unproven, but believed that there are infinitely many amicable pairs. The distribution of amicable pairs within the natural numbers has been studied, and upper bounds have been established for $A(x)$, the number of amicable numbers in $[1, x]$. Amicable pairs have also been studied under the view of function iteration. An *aliquot sequence* is a sequence of positive integers where each term is the sum of the proper divisors of the previous term. An amicable pair is an aliquot sequence with a cycle of length two, *sociable numbers* are cycles of length greater than two, and a *perfect number* has a repeating aliquot cycle of length one. The Catalan-Dickson conjecture says that all aliquot sequences either terminate or are eventually periodic.

The complex sum of divisors function was introduced by R. Spira as a natural extension of the standard sum of divisors function on the real numbers to the Gaussian integers in [20]. Using it, we can study how many properties characterized by the sum of divisors function extend to the Gaussian integers. In [12] we extended the study of amicable pairs to the complex numbers. We answered the open question of whether Gaussian amicable pairs existed. We found over 100 new Gaussian amicable pairs with a complex part using *Mathematica*. The crux for finding these pairs was the short command:

$$\begin{aligned} m &= \text{DivisorSigma}[1, n, \text{GaussianIntegers} \to \text{True}] - n; \\ x &= \text{DivisorSigma}[1, m, \text{GaussianIntegers} \to \text{True}] - m; \\ \text{If } &[x == n, \text{Print}[m, \text{" and "}, n \text{" are amicable "}]] \end{aligned}$$

One search in connection with this test was to use nested loops that created m values as $m = a + bi$, where a ranged from 0 to a large integer and b ranged from 0 to a large integer. We also identified classes of amicable pairs in the natural numbers that are also amicable when viewed as Gaussian integers and showed there are no (2,1) pairs in the integers, such as (220,284), that are also Gaussian amicable pairs.

3 Mathematics Education

My research interests in mathematics education include online curriculum design, flipped instruction, service learning, inclusive pedagogy, and community engagement. At the University of Iowa, I completed a certificate program in online teaching, where I learned to use Universal Design for Learning (UDL) principles as a framework for course design. The three overarching principles include providing multiple means of engagement, representation, and action and expression in the creation of challenging learning opportunities and environments with accessibility in mind to benefit all learners. These principals are used in curriculum design to provide multiple ways to address the affective, strategic, and recognition networks that focus on the why, the how, and the what of learning respectively. I applied these principles to a Teaching as Research Project through the Center for the Integration for Research Teaching and Learning (CIRTL) at the University of Iowa, where I studied how partially flipping my trigonometry course affected my students perceptions of their own learning. I also applied this framework to the curriculum design of a service learning course at Ohio State.

In 2019, I co-developed the first service learning course offered by the Department of Mathematics at The Ohio State University entitled, *Intersections of Mathematics and Society: Hidden Figures* which focused on inclusivity in mathematics. In this course, we directly addressed the connection between the creation of mathematics, its developments and applications, and society. The course featured many instances of embedded reflection, students participated in service learning and outreach, and we were intentional about highlighting external voices. These particular design decisions are rooted in the three pillars of culturally competent pedagogy: (1) academic achievement, (2) cultural competence, and (3) sociopolitical consciousness. An article [13] about the course and on these instructional practices will appear in the Early Career Section of the Notices of the American Mathematical Society in 2021.

Work on this course led to an interest in the 'hidden' stories of Black mathematicians at Ohio State. I have discovered nearly 200 Black mathematicians that have matriculated through the Department of Mathematics at The Ohio State University, some of whom have gone on to become prolific researchers, lawyers, high school teachers, economists, and university presidents, yet their legacies are largely 'hidden.' To remedy this, I have initiated a collaboration between faculty and staff across various colleges and offices at Ohio State, the National Math Alliance, and the National Afro-American Museum and Cultural Center (NAAMCC) of the Ohio History Connection to develop a case study on Ohio State's Black mathematicians. I am the PI for an internal grant that has advanced to the final round of funding which will fund an interdisciplinary research team of students from mathematics, english, communication, and photography to develop a digital archive of Black mathematicians from our case study research. NAAMCC will integrate our archive into their digital collections, and will work alongside our team to develop an 8th-12th grade teacher resource package as part of their initiative to connect their history and social studies resources to STEM. This effort will be the first comprehensive study of Black mathematicians at a single US institution.

4 Undergraduate Research

Numerical Semigroup Algebras Numerical semigroups are a vibrant area of undergraduate research, and problems in this area have low floors but high ceilings, making them accessible to students with a wide variety of backgrounds. Many problems benefit from computational inquiry, providing appeal to students with interests in mathematics, statistics, and computer science. In [9] we used the GAP package numericalsgps from within Sage with the help of the numsgpsalg [18] and NumericalSemigroup.sage [17] packages, both available on Github. This aidied in allowing us to characterize irreducibles in $\mathbb{F}_2[x^2, x^3]$ into three distinct classes which led to our main result. This characterization relied on understanding the factorization elements of $\mathbb{F}_q[S]$ as elements of $\mathbb{F}_q[x]$, but did not pay attention to forms of irreducibles in $\mathbb{F}_q[x]$ aside from a brief remark in [9] about irreducibles of the form $x^p + 1$ in $\mathbb{F}_q[S]$. The following would serve as the basis for a thesis for a student who has taken modern algebra and possibly number theory: **(1)** Investigate the forms of irreducibles in $\mathbb{F}_q[S]$ for a a particular prime q and numerical semigroup S.

Amicable Pairs In Section 2 I highlighted past work on Gaussian Amicable Pairs [12]. Recall that *Amicable pairs*, or *friendly numbers*, in the natural numbers are a pair of distinct positive integers (m, n) where each integer is the sum of the proper divisors of the other integer. Many research problems involving amicable numbers are related to questions of existence and to the distribution of certain types of pairs, providing avenues for algebraic and statistical thinking within elementary number theory. This research program is ideal for students with a variety of course preparation ranging from discrete mathematics to number theory to modern algebra. The following questions are some examples of questions that could form the basis for undergraduate research projects: **(1)** Do there exist Gaussian aliquot sequences of length greater than 2? How can we realize such sequences with a directed graph? What do cycles and loops in such a graph represent? **(2)** Can we define Gaussian superperfect numbers? Gaussian norm-superperfect numbers? What can be said about odd numbers of these types? **(3)** Can we extend the notion of amicable pairs to other Euclidean domains? **(4)** Find new Gaussian amicable pairs of a certain type using mother daughter pairs and other techniques.

Math Education The course *Intersections of Mathematics and Society: Hidden Figures* features a unit where students learn how to operate slide rules, computing devices utilizing a logarithmic scale to perform operations ranging from multiplication and division to hyperbolic trigonometry. A mathematics education student could work on the following related project: **(1)** Fostering Conceptual Understanding of Logarithms Using Slide Rules. The pedagogical framework for this course also lends itself to other projects that could be explored with education majors. The following are sample project ideas: **(2)** Explore the utility of service learning as a mechanism for student exposure to career options outside of academia. **(3)** Explore the use of discussion boards to improve mathematical reasoning and communication, as well as to facilitate community in online and in-person environments.

The case study research on Black mathematicians at The Ohio State University could also be used to spearhead student projects. The first Black mathematician to earn a PhD from The Ohio State University, William McWorter Jr. wrote several papers describing determinant free algorithms to compute eigenvectors and the characteristic polynomial of an $n \times n$ matrix with real entries. There has been an expressed interest in shaping these ideas into lesson plans or projects that can be made into open source materials for use by instructors interested in challenging and culturally competent resources in matrix algebra. The following project ideas would only require a linear algebra background: **(4)** Investigate Determinant Free Methods in Linear Algebra **(5)** Create lesson plans or projects related to determinant free methods to compute the characteristic polynomial and eigenvectors of a matrix.

A goal of the case study research is to make a digital archive of Black mathematicians. This practice of researching historical figures and their contributions, and then creating digital artifacts highlighting this work has interdisciplinary potential with other areas in the arts and sciences. Consider the following related project: **(6)** Use Omeka digital archive software to highlight historical contributions of underrepresented scholars in STEM using dynamic and visual storytelling.

5 Future Work

I continue to work on questions in algebra related to factorization in commutative settings as highlighted above in section 2, as well problems related to mathematics education which are noted in section 3. In the long term, I aim to expand my research interests to include homological algebra and algebraic topology. My motivations for this are to (1) increase the scope of problems I work on within commutative algebra (2) broaden my toolbox to include techniques in applied algebra, namely through a focus on applications of algebraic topology and homological algebra, and (3) to set the foundation for a robust research program for work with undergraduate students with interests in pure and applied algebra.

During the 2020-2021 academic year I am being supported by NSF RTG grant Topology and its Applications # 1547357. During this time period I aim to broaden my research program in algebra by focusing on two specific directions. The first is the expansion of my work with numerical semigroups to general semigroup theory, and in particular its intersections with homological algebra. The second is applications of persistent homology within topological data analysis. As part of the RTG, in the Spring of 2021 I will teach the undergraduate Introduction to Applied Algebraic Topology course at The Ohio State. This course requires students to have taken linear algebra, along with discrete math or a comparable introduction to proofs course. It focuses on introducing students to the area of topological data analysis with a view toward persistent homology of point cloud data. I will design a project for the students that allows them to use topological techniques to analyze real data sets. These projects could lead to independent research projects or readings to further study advanced topics from the course.

I am in the early stages of a project [15] that lies at the intersection of commutative semigroups and homological algebra. This new project is part of a broader effort to extend homological algebra for semigroups. Our main goal is to create a Dold-Kan type theorem for commutative inverse semigroups. The fundamental Dold-Kan theorem gives a categorical equivalence between connective chain complexes and simplicial abelian groups. This correspondence tells us that that the n^{th} homotopy group of a simplicial abelian group is precisely the n^{th} homology group of the corresponding chain complex, thus allowing for the use of homological algebra techniques to study simplicial homotopy. We will explore whether there exists a model equivalence between the category of commutative inverse semigroups and some other category of objects that slightly generalizes a connective chain complex. This is motivated by the desire to understand whether taking the free commutative simplicial monoid $N[X]$ of some simplicial set X, gives us similar data as a chain complex. This investigation can lead to definitions of homology for directed topology which would be of great interest to semigroup and homotopy theorists.

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